AN EXTREMAL PROBLEM IN GRAPH THEORY

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ABSTRACT

It is proved that the maximum number of cut-vertices in a connected graph with n vertices and m edges is

$$\max\left\{q: m \leqq \binom{n-q}{2} + q\right\}.$$

All the extremal graphs are determined and the corresponding problem for cut-edges is also solved.

In this paper we determine the maximum number of cut-vertices in a connected graph on n vertices with m edges and also the class of all extremal graphs, i.e., graphs which attain this maximum. The analogous problem for cut-edges is also solved.

All graphs considered here are finite, undirected and without multiple edges or loops.

For notation and terminology C. Berge [1] is followed.

We will use the term cut-vertex (edge) for an articulation vertex (edge) of [1].

A *block* of a connected graph G is a subgraph of G which is maximal with respect to the property of being connected and having no cut-vertex.

§1. Maximisation of the number of cut-vertices.

We start with a few preliminary results.

LEMMA 1.1. *A connected graph on n vertices ($n \geqq 2$) has at most $n-2$ cut-vertices. Further, the only such graph with n–2 cut-vertices is the elementary chain on n vertices.*

This lemma can be easily proved by using the concept of a spanning tree.

THEOREM 1.2. *In a connected graph on n vertices with r cut-vertices, the maximum number of edges is* $\binom{n-r}{2} + r$.

Proof. Let G be a connected graph on n vertices with r cut-vertices and with the maximum number of edges. Then obviously every block of G is complete and the number t of blocks is not less than $r+1$. Let n_i be the number of vertices in the ith block for $i = 1, 2, \cdots, t$. Then $n_i \geqq 2$ and it can be easily proved by induction on t that

$$\sum_{i=1}^{t} n_i = n + t - 1.$$

Received May 26, 1967, and in revised form May 17, 1968.

Thus the number of edges in G is not more than

$$\max\left\{\sum_{i=1}^{t}\binom{n_i}{2} : \sum_{i=1}^{t} n_i = n+t-1,\ n_i \geqq 2,\ t \geqq r+1\right\}$$

$$= \max_{t \geqq r+1}\left\{t-1+\binom{n+t-1-2t+2}{2}\right\}$$

$$= \binom{n-r}{2}+r.$$

But a complete graph on $n-r$ vertices with an attached elementary chain of length r has n vertices, $\binom{n-r}{2}+r$ edges and r cut-vertices.

This completes the proof of the theorem.

Given n, m such that $n \leqq m \leqq \binom{n}{2}$, let us define

$$r(n,m) = \max\left\{q : q \leqq n-3 \text{ and } m \leqq \binom{n-q}{2}+q\right\}. \tag{1.1}$$

We assume below that $m \geqq n$ as the case $m = n-1$ is trivial and is treated completely in Lemma 1.1.

THEOREM 1.3. *The maximum number of cut-vertices in a connected graph on n vertices with m edges is* $r = r(n,m)$ *given by* (1.1).

Proof. By Theorem 1.2, if a connected graph G on n vertices has $r+1$ or more cut-vertices then the number of edges in G is not more than

$$\binom{n-r-1}{2}+r+1 < m.$$

Hence the number of cut-vertices in any connected graph on n vertices with m edges is not more than r. To construct a connected graph with n vertices, m edges and exactly r cut-vertices, take any biconnected graph on $n-r$ vertices with $m-r$ edges

$$\left(n-r \leqq m-r \leqq \binom{n-r}{2}\right),$$

and attach to one of its vertices an elementary chain of length r. This proves the theorem.

Now we determine the extremal graphs, i.e., connected graphs on n vertices with m edges and with $r(n,m)$ cut-vertices.

LEMMA 1.4. *In an extremal graph G there cannot be more than 2 pieces with respect to any cut-vertex.*

Proof. If there are at least 3 pieces with respect to a cut-vertex x of G, we can remove one of the pieces (without the vertex x itself) and attach it at a non-cut-vertex of the remaining graph, thereby increasing the number of cut-vertices. The impossibility of this proves the lemma.

LEMMA 1.5. *The graph G consisting of two complete graphs, each on more than 3 vertices, attached by a common vertex, is not extremal.*

Proof. Let r, s be the numbers of vertices in the two complete subgraphs of G so that the number n of vertices in G is $r + s - 1$. Then the number of edges in G is not more than

$$\max_{\substack{r,s \geqq 4,\\ r+s=n+1}} \left\{\binom{r}{2} + \binom{s}{2}\right\}$$

$$= \binom{4}{2} + \binom{n-3}{2} = \binom{n-2}{2} + 2 + 7 - n.$$

Hence if G is extremal, then by Theorem 1.3 G has at least 2 cut-vertices, a contradiction.

LEMMA 1.6. *An extremal graph G without any cut-edge has at most one cut-vertex.*

Proof. Adding new edges if necessary we make every block of G complete. The resulting graph H is also extremal since it has the same number of cut-vertices as G, say r, but more edges. By Lemma 1.4, the number of blocks in H is $r + 1$. If n_i is the number of vertices in the ith block, then $n_i \geqq 3$, since H has no cut-edge. Thus the number of edges in H is not more than

$$\max \left\{\sum_{i=1}^{r+1} \binom{n_i}{2} : \sum_{i=1}^{r+1} n_i = n + r,\ n_i \geqq 3\right\}$$

$$= 3r + \binom{n-2r}{2}.$$

The right hand expression is not greater than

$$r + 1 + \binom{n-r-1}{2}$$

whenever $r \geqq 2$. But H has only r cut-vertices and this gives a contradiction to Theorem 1.3. Thus there is at most one cut-vertex in an extremal graph without any cut-edge.

LEMMA 1.7. *If an extremal graph G with n vertices and m edges has no cut-edge, then either*

(1) $m \geqq \binom{n-1}{2} + 2$

or

(2) *G consists of a complete graph and a triangle attached to it by a common vertex.*

Proof. Suppose (1) does not hold. Then G has exactly one cut-vertex by Lemma 1.6. Now making each block of G complete we get a graph consisting of two complete subgraphs attached by a common vertex. By Lemma 1.5, at least one of these complete subgraphs is a triangle. If the other block is not complete in G, then we can transfer one of the edges of the triangle to it, thereby increasing the number of cut-vertices. The impossibility of this proves the lemma.

Now we will prove the main result of this section. Let $r = r(n, m)$ be given by (1.1).

THEOREM 1.8. *The extremal graphs on n vertices with m edges are the following:*

(1) *a graph consisting of a subgraph on n_0 vertices with m_0 edges to which elementary chains of total length r are attached at distinct vertices, where* $n_0 = n - r$ and $m_0 = m - r \geqq \binom{n_0 - 1}{2} + 2.$

(2) *a graph consisting of an elementary chain μ (which may be a single vertex) separating a complete graph at one end and a triangle at the other end, with elementary chains attached at distinct vertices not belonging to μ, where the sum of the lengths of μ and all the terminal chains is $r - 1$.*

Proof. Let G be an extremal graph on n vertices with m edges. By successively removing a pendant vertex and its incident edge, we finally arrive at a subgraph H without pendant vertices such that G is obtained from H by attaching trees at some of the vertices of H. Each of these trees is an elementary chain, for otherwise we can increase the number of cut-vertices by replacing such a tree by a chain on the same number of vertices. Evidently now H is also extremal. Let n_0, m_0 be the number of vertices and the number of edges respectively in H.

If H has no cut-vertex, then obviously G is of the type (1) of the theorem.

If H has a cut-vertex, then there is a unique elementary chain (which may be a single vertex) separating blocks on more than 2 vertices. For otherwise, by suppressing every such chain and identifying its end vertices we get an extremal graph without any cut-edge and with at least two cut-vertices, which is a contradiction to Lemma 1.6. By the same argument it follows from Lemma 1.7 that H consists of an elementary chain μ separating a complete graph at one end and a triangle at the other end. Obviously now G is of the type (2) of the theorem.

It is trivial to see that the minimum number of cut-vertices in a connected graph on n vertices with m edges is 0 or 1 according as $m \geqq n$ or $m = n - 1$.

§2. Maximisation of the number of cut-edges.

LEMMA 2.1. *The maximum number of cut-edges in a connected graph on n vertices is $n-1$. This maximum is attained by any tree and by no other graph.*

This lemma can be easily proved using the concept of a spanning tree.

THEOREM 2.2. *If $r \leqq n-2$ then the number of edges in a connected graph G on n vertices with r cut-edges is not more than $\binom{n-r}{2}+r$.*

Proof. Since $r \neq n-1$, G has at least one cycle and hence there are at least $r+1$ blocks, namely the r cut-edges and another block on at least 3 vertices. Now the proof of the lemma is similar to that of Theorem 1.2.

Given n, m such that $n \leqq m \leqq \binom{n}{2}$, let $r = r(n,m)$ be given by (1.1). We assume below that $m \geqq n$ since the case $m = n-1$ is completely treated in Lemma 2.1.

THEOREM 2.3. *The maximum number of cut-edges in a connected graph on n vertices with m edges is r.*

The proof of this theorem utilizes Theorem 2.2 and is similar to that of Theorem 1.3.

We call a graph which attains the maximum number r of cut-edges an 'extremal' graph.

THEOREM 2.4. *Any 'extremal' graph G on n vertices with m edges consists of a subgraph H on n_0 vertices and m_0 edges to which trees with a total of r edges are attached at some vertices, where $n_0 = n - r$, $m_0 = m - r \geqq \binom{n_0-1}{2}+2$. (The converse is obvious.)*

Proof. As shown in the proof of Theorem 1.8, there exists a subgraph H without pendant vertices such that G is obtained from H by attaching trees at some of the vertices of H. Obviously now H is also 'extremal'. If H has a cut-edge, then by successively suppressing such edges and identifying their end vertices we get an 'extremal' graph without cut-edges and with at least one cut-vertex. If this graph has p vertices, then by Theorem 2.3 it has at least $\binom{p-1}{2}+2$ edges and therefore does not have any cut-vertex. This contradiction shows that if n_0, m_0 are the number of vertices and the number of edges of H respectively, then $n_0 = n - r$ and $m_0 = m - r \geqq \binom{n-r}{2}+r$. This completes the proof of the theorem.

It is easy to see that the minimum number of cut-edges in a connected graph on n vertices with m edges is 0 or $n-1$ according as $m \geqq n$ or $m = n-1$.

I wish to thank Dr. C. Ramanujacharyulu for the many useful discussions I had with him about the contents of this paper. I am also thankful to Dr. U.S.R. Murty for suggesting this problem to me and to the referee for his comments on an earlier version of this paper.

Reference

1. C. Berge, *The Theory of Graphs and its Applications*, Methuen, London, 1962.

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